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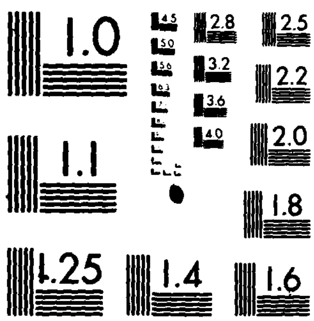

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DECOMPOSITIONS OF MULTIATTRIBUTE UTILITY FUNCTIONS BASED ON CONVEX DEPENDENCE

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~~We~~ describes a method of assessing multiattribute utility functions. ~~First we introduce~~ the concept of convex dependence, ~~where we~~ consider the change of shapes of conditional utility functions. Then, ~~we establish~~ theorems which show how to decompose multiattribute utility functions using convex dependence. The convex decomposition includes as special cases Keeney's additive/multiplicative decompositions, Fishburn's bilateral decomposition, and Bell's decomposition under the interpolation independence. Moreover, the convex decomposition is an exact grid model which was axiomatized by Fishburn and Farquhar.

DECOMPOSITIONS OF MULTIATTRIBUTE UTILITY FUNCTIONS BASED ON CONVEX DEPENDENCE

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ABSTRACT

We describe a method of assessing von Neumann-Morgenstern utility functions on a two-attribute space and its extension to n-attribute spaces. First, we introduce the concept of convex dependence between two attributes, where we consider the change of shapes of conditional utility functions. Then, we establish theorems which show how to decompose a two-attribute utility function using the concept of convex dependence. This concept covers a wide range of situations involving trade-offs. The convex decomposition includes as special cases Keeney's additive/multiplicative decompositions, Fishburn's bilateral decomposition, and Bell's decomposition under the interpolation independence. Moreover, the convex decomposition is an exact grid model which was axiomatized by Fishburn and Farquhar. Finally, we extend the convex decomposition theorem from two attributes to an arbitrary number of attributes.

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This paper deals with individual decision making where the decision alternatives are characterized by multiple attributes. The problem is to provide conditions describing how a decision maker trades off conflicting attributes in evaluating decision alternatives. These conditions then restrict the form of a multiattribute utility function in a decomposition theorem. In many situations, it is practically impossible to directly assess a multiattribute utility function, so it is necessary to develop conditions that reduce the dimensionality of the functions that are required in the decomposition.

Much of the research in utility theory deals with additive decompositions [5, 16]. Pollak [16], Keeney [11, 12, 13, 14], and others, however, develop a "utility independence" condition that implies non-additive utility decompositions. Although these decompositions have been applied to many real-world decision problems, there are situations, such as conflict resolution between pollution and consumption [17], where the utility independence condition does not hold. Fishburn [6] and Farquhar [3, 4] have investigated more general independence conditions that imply various non-additive utility decompositions. For example, Farquhar's fractional decompositions include nonseparable attribute interactions.

In this paper, we introduce the concept of convex dependence as an extension of utility independence. In our methodology, normalized conditional utility functions play an important role. Utility independence implies that the normalized conditional utility functions do not depend on different conditional levels. On the other hand, convex dependence implies that each normalized conditional utility functions can be represented as a convex combination of some specified normalized conditional utility functions. Keeney [12] described interpolation in motivating utility independence. If we find that

the utility independence condition does not hold in the process of assessing normalized conditional utility functions, we can repeat the procedure [17] to test the convex dependence condition to derive the utility representations as approximations. the concept of the convex dependence covers a wide range of situations involving trade-offs. The convex decomposition includes as special cases Keeney's [12, 13] multilinear and multiplicative decompositions, Fishburn's [6] bilateral decomposition, and Bell's [1] decomposition under interpolation independence, which is the same as first-order convex dependence in this paper. Bell [2] has developed ways to reduce the number of constants to be assessed and has provided a generalization of additive and multiplicative forms in the multiattribute case. Moreover, the convex decomposition is an exact grid model as defined by Fishburn [7]. Our approach gives an approximation of utility functions but recently Fishburn and Farquhar [8] derived a preference axiom which provides a general exact grid model, and provided a procedure for selecting the normalized conditional utility functions.

1. PRELIMINARIES

Let $X = X_1 \times \ldots \times X_n$ denote the consequence space which, for simplicity, is a rectagular subset of a finite-dimensional Euclidean space. A specific consequence $x \in X$ is represented by $(x_1, \ldots, x_n)$, where x_i is a particular level in the attribute set X_i. We consider $Y \times Z$ as two-attribute space, where $Y = X_{i_1} \times \ldots \times X_{i_r}$, $Z = X_{i_{r+1}} \times \ldots \times X_{i_n}$ and $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. Throughout the paper, we assume that appropriate conditions are satisfied for the existence of von Neumann-Morgenstern utility function $u(y,z)$ on $Y \times Z$ [18]. Moreover, we assume that there exist distinct y^*, $y^0 \in Y$

which satisfy $u(y^*,z) \neq u(y^0,z)$ for all $z\varepsilon Z$. Similarly, we assume that there exist distinct $z^*,z^0 \varepsilon Z$, which satisfy $u(y,z^*) \neq u(y,z^0)$ for all $y\varepsilon Y$.

DEFINITION 1. Given an arbitrary $z\varepsilon Z$, a *normalized conditional utility function* $v_z(y)$ on Y is defined as

$$v_z(y) = \frac{u(y,z) - u(y^0,z)}{u(y^*,z) - u(y^0,z)}.$$

From Definition 1 it is obvious that $v_z(y^0) = 0$ and $v_z(y^*) = 1$. Moreover, if a decision maker prefers y^* to y^0, then $v_z(y)$ represents his utility, and if a decision maker prefers y^0 to y^*, then $v_z(y)$ represents his disutility.

To represent the decomposition forms and proofs simply, we need to introduce some notation. First, we define three functions $f(y,z)$, $G(y,z)$ and $H(y,z)$ which will be used to represent the decompostion forms. We assume $u(y^0,z^0) \equiv 0$ without loss of generality.

$$f(y,z) \equiv u(y,z) - u(y^0,z) - u(y,z^0), \tag{1}$$

$$G(y,z) \equiv u(y^*,z^0)f(y,z) - u(y,z^0)f(y^*,z), \tag{2}$$

$$H(y,z) \equiv u(y^0,z^*)f(y,z) - u(y^0,z)f(y,z^*). \tag{3}$$

The two functions $G(y,z)$ and $H(y,z)$ are related to each other as follows.

$$u(y^0,z^*)G(y,z) - u(y^0,z)G(y,z^*) = u(y^*,z^0)H(y,z) - u(y,z^0)H(y^*,z). \tag{4}$$

We define $F(y,z)$ as

$$F(y,z) \equiv u(y^0,z^*)G(y,z) - u(y^0,z)G(y,z^*). \tag{5}$$

To represent the constants simply in our decomposition forms, three matrices G^n H^n and F^n are defined for $y^1, \ldots, y^n \varepsilon Y$ and $z^1, \ldots, z^n \varepsilon Z$. Let the (i,j)

element of the matrix G^n be denoted by $(G^n)_{ij}$, which is defined as $G(y^i,z^j)$, where $z^n \equiv z^*$. Similarly, define $(H^n)_{ij} \equiv H(y^j,z^i)$, where $y^n \equiv z^*$, and define $(F^n)_{ij} \equiv f(y^j,z^i)$, where $y^n = y^*$ and $z^n = z^*$. Let G^n_{ij} be the $(n-1) \times (n-1)$ matrix obtained from G^n by deleting the i-th row and the j-th column, and let "det" denote the determinant on square matrices. Define

$$|G^n| = \det(G^n),\ \tilde{G}^n_{ij} = (-1)^{i+j}|G^n_{ij}|,\ i,j = 1, \ldots, n.$$

Let $|H^n|$, $\tilde{H}^n_{ij}$, $|F^n|$ and $\tilde{F}^n_{ij}$ be defined similarly. Moreover, for $n = 1$, we define $\tilde{G}^1_{ij} = \tilde{H}^1_{ij} = \tilde{F}^1_{ij} = 1$. We define an $n \times n$ matrix G_n for distinct $y_1, \ldots, y_n \varepsilon Y$, and distinct $z_0, z_1, \ldots, z_n \varepsilon Z$ as $(G_n)_{ij} = v_{z_j}(y_i) - v_{z_0}(y_i)$.

2. CONVEX DEPENDENCE AND ITS PROPERTIES

In this section, we define the concept of convex dependence and discuss some of its properties. In the following, let δ_{ij} be the Kronecker delta function.

DEFINITION 2. Y is *n-th order convex dependent* on Z, denoted $Y(CD_n)Z$, if there exist distinct $z_0, z_1, \ldots, z_n \varepsilon Z$ and real functions $g_1, \ldots, g_n$ on Z with $g_i(z_j) = \delta_{ij}$ for $i\varepsilon\{1, \ldots, n\}$ and $j\varepsilon\{0, 1, \ldots, n\}$ such that the normalized conditional utility function $v_z(y)$ can be written as

$$v_z(y) = [1 - \sum_{i=1}^{n} g_i(z)]\, v_{z_0}(y) + \sum_{i=1}^{n} g_i(z)\, v_{z_i}(y) \qquad (6)$$

for all $y\varepsilon Y$ and $z\varepsilon Z$, where n is the smallest non-negative integer for which (6) holds.

For $n = 1$, relation (6) implies "Y is interpolation independent of Z" in Bell's [1, 2] terminology. When Y and Z are scalar attributes, a geometric illustration of Definition 2 is in Figure 1. Suppose three arbitrary normalized conditional utility functions $v_{z_0}(y)$, $v_{z_1}(y)$, and $v_z(y)$ are assessed on Y. If $Y(CD_0)Z$, all the normalized conditional utility functions are identical as shown in Figure 1(a). If $Y(CD_1)Z$, an arbitrary normalized conditional utility function $v_z(y)$ can be obtained as a convex combination of $v_{z_0}(y)$ and $v_{z_1}(y)$ as shown in Figure 1(b). Moreover, Figure 1(b) shows that the preferential independence condition [9] need not hold (Note that $v_{z_0}(y)$ is monotonic and $v_{z_1}(y)$ is not.).

Figure 1 goes here

We now establish several properties of convex dependence. Let Y(GUI)Z denote Y is generalized utility independent of Z: see Fishburn and Keeney [10] for a definition.

PROPERTY 1. $Y(CD_0)Z$, if and only if Y(GUI)Z.

Proof. If Y(GUI)Z, the following equation holds

$$u(y,z) = \alpha(z)u(y,z_0) + \beta(z) \tag{7}$$

for some $z_0 \in Z$. Setting $y = y^0$ and $y = y^*$ in (7) where $u(y^0,z) \neq u(y^*,z)$ for all $z \in Z$ by the assumption in section 1, we obtain

$$u(y^0,z) = \alpha(z)u(y^0,z_0) + \beta(z), \tag{8a}$$

$$u(y^*,z) = \alpha(z)u(y^*,z_0) + \beta(z). \tag{8b}$$

Therefore,

$$\frac{u(y,z) - u(y^0,z)}{u(y^*,z) - u(y^0,z)} = \frac{\alpha(z)[u(y,z_0) - u(y^0,z_0)]}{\alpha(z)[u(y^*,z_0) - u(y^0,z_0)]} = \frac{u(y,z_0) - u(y^0,z_0)}{u(y^*,z_0) - u(y^0,z_0)}. \quad (9)$$

From the Definition 1, (9) implies that $v_z(y) = v_{z_0}(y)$ which shows that $Y(CD_0)Z$.

If $Y(CD_0)Z$, (9) holds. Rearranging (9), we obtain

$$u(y,z) = \frac{u(y^*,z) - u(y^0,z)}{u(y^*,z_0) - u(y^0,z_0)}\, u(y,z_0) + \frac{u(y^0,z)u(y^*,z_0) - u(y^0,z_0)u(y^*,z)}{u(y^*,z_0) - u(y^0,z_0)} \quad (10)$$

which shows that $Y(GUI)Z$. ■

This property shows that the convex dependence is a natural extension of generalized utility independence except for null zones.

PROPERTY 2. If $Y(CD_n)Z$, then there exist distinct $y_1, \ldots, y_n \varepsilon Y$, and distinct $z_0, z_1, \ldots, z_n \varepsilon Z$ which satisfy rank $G_n = n$.

Proof. On the contrary, suppose rank $G_n \neq n$ for all distinct $y_1, \ldots, y_n \varepsilon Y$ and $z_0, z_1, \ldots, z_n \varepsilon Z$. Then there exist real numbers h_i ($i = 1, \ldots, n$) such that for all $y \varepsilon Y$, we have

$$v_{z_n}(y) - v_{z_0}(y) = \sum_{i=1}^{n-1} h_i [v_{z_i}(y) - v_{z_0}(y)]$$

which implies $Y(CD_{n-1})Z$. ■

Using Property 2, we can assess the order of convex dependence [17]. For $n = 1, 2, \ldots$ sequentially we test the rank condition of G_n for arbitrary distinct $y_1, \ldots, y_n \varepsilon Y$. Then if rank $G_n = n$ and rank $G_{n+1} = n$ for arbitrary distinct $y_1, \ldots, y_{n+1} \varepsilon Y$, we can conclude $Y(CD_n)Z$.

It is obvious that relation between G_n and G^n is as follows

$$\text{rank } G_n = \text{rank } G^n$$

for distinct $y^1, \ldots, y^n \varepsilon Y$ and distinct $z^0, z^1, \ldots, z^{n-1}, z^* \varepsilon Z$, because $G(y,z) = u(y^*,z^0)[u(y^*,z) - u(y^0,z)][v_z(y) - v_{z^0}(y)]$ from (1) and (2). Thus we immediately get the following property.

PROPERTY 3. If $Y(CD_n)Z$, then there exist distinct $y^1, \ldots, y^n \varepsilon Y$ and distinct $z^1, \ldots, z^{n-1} \varepsilon Z$ which satisfy rank $G^n = n$.

Obviously the same property of rank condition for H^n holds. Property 3 guarantees that the following property holds, which shows the relation of the order of convex dependence between two attributes.

PROPERTY 4. For $n = 0, 1, \ldots$, if $Y(CD_n)Z$, then Z is at most $(n + 1)$-th order convex dependent on Y.

Proof. See appendix.

A few aspects of these Properties deserve brief comment. If Y is utility independent of Z which is denoted Y(UI)Z, then Y is obviously convex dependent on Z; the converse is not true. The concept of convex dependence asserts that when Y is utility independent of Z, Z must be utility independent or first-order convex dependent on Y. Moreover, if Y is n-th order convex dependent on Z, then Z satisfies one of the three properties, $Z(CD_{n-1})Y$, $Z(CD_n)Y$, or $Z(CD_{n+1})Y$, because if $Z(CD_m)Y$ for $m < n - 1$, then $Y(CD_{m+1})Z$ at most and $m + 1 < n$.

PROPERTY 5. If rank $G^n = n$ for distinct $y^1, \ldots, y^n \epsilon Y$ and distinct $z^1, \ldots, z^{n-1} \epsilon Z$, then rank $F^n = n$.

Proof. By using (2), we obtain the following relation between G^n and F^n.

$$|G^n| = [u(y^*,z^0)]^{n-1} \{ \sum_{i=1}^{n^*} u(y^i,z^0) \sum_{j=1}^{n^*} \tilde{F}_{ji} f(y^n,z^j) - u(y^n,z^0)|F^n| \},$$

where summation $i = 1$ to n^* means $i = 1, 2, \ldots, n-1, *$.
On the contrary, if rank $F^n \neq n$ for distinct $y^1, \ldots, y^{n-1} \epsilon Y$ and $z^1, \ldots, z^{n-1} \epsilon Z$, then,

$$|F^n| = 0 \text{ and } \sum_{j=1}^{n^*} \tilde{F}^n_{ji} f(y^n,z^j) = 0 \text{ for } i = 1, 2, \ldots, n$$

because even if we transform one of $y^1, y^2, \ldots, y^{n-1}$ and y^* into y^n in F^n, rank $F^n \neq n$ by the assumption. ■

3. CONVEX DECOMPOSITION THEOREMS ON TWO-ATTRIBUTE SPACE

This section uses convex dependence to establish two decomposition theorems and a corollary for two-attribute utility functions. We further discuss the relation of these results with the previous researches.

THEOREM 1. For $n = 1, 2, \ldots, Y(CD_n)Z$, if and only if

$$u(y,z) = u(y^0,z) + u(y,z^0) + v(y)f(y^*,z) + \frac{c_y}{|G^n|} \sum_{i=1}^{n^*} \sum_{j=1}^{n} \tilde{G}^n_{ji} G(y,z^i)G(y^j,z), \quad (11)$$

where $\quad v(y) = \dfrac{u(y,z^0)}{u(y^*,z^0)}, \; c_y = \dfrac{1}{u(y^*,z^0)}.$

Proof. See appendix.

THEOREM 2. For $n = 1, 2, \ldots$, $Y(CD_n)Z$ and $Z(CD_n)Y$, if and only if

$$u(y,z) = u(y^0,z) + u(y,z^0) + \frac{1}{|F^n|}\sum_{i=1}^{n^*}\sum_{j=1}^{n^*}\tilde{F}^n_{ij}f(y,z^i)f(y^j,z)$$

$$+ c\sum_{i=1}^{n^*}\sum_{j=1}^{n^*}\tilde{G}^n_{ni}\tilde{H}^n_{nj}G(y,z^i)H(y^j,z), \tag{12}$$

where $c = \frac{c_y c_z}{|G^n H^n|}\left[f(y^n,z^n) - \frac{1}{|F^n|}\sum_{i=1}^{n^*}\sum_{j=1}^{n^*}\tilde{F}^n_{ij}f(y^n,z^i)f(y^j,z^n)\right]$

and $c_y = \frac{1}{u(y^*,z^0)}$, $c_z = \frac{1}{u(y^0,z^*)}$.

Proof. See appendix.

We have obtained two main decomposition theorems which can represent a wide range of utility functions. Moreover, when the utility on the arbitrary point (y^n,z^n) has a particular value, that is, $c = 0$ in (12), we can obtain one more decomposition of utility functions which does not depend on the point (y^n,z^n). This decomposition still satisfies $Y(CD_n)Z$ and $Z(CD_n)Y$, so we will call this new property *reduced n-th order convex dependence* and denote it by $Y(RCD_n)Z$. It is obvious that $Z(RCD_n)Y$ when $Y(RCD_n)Z$.

COROLLARY 1. For $n = 1, 2, \ldots$, $Y(RCD_n)Z$, if and only if

$$u(y,z) = u(y^0,z) + u(y,z^0) + \frac{1}{|F^n|}\sum_{i=1}^{n^*}\sum_{j=1}^{n^*}\tilde{F}^n_{ij}f(y,z^i)f(y^j,z). \tag{13}$$

We note that when $n = 1$, (13) reduces to Fishburn's [6] bilateral decomposition,

$$u(y,z) = u(y^0,z) + u(y,z^0) + \frac{f(y,z^*)f(y^*,z)}{f(y^*,z^*)}. \tag{14}$$

In Figure 2, we show on two scalar attributes the difference between the conditional utility functions necessary to construct the previous decomposition models and our decomposition models. By assessing utilities on the heavy shaded lines and points, we can completely specify the utility function in the cases indicated in Figure 2. As seen from Figure 2, an advantage of the convex decomposition is that only conditional utility functions with one varying attribute need be assessed even for high-order convex dependent cases.

Figure 2 goes here

4. CONVEX DECOMPOSITION THEOREM ON N-ATTRIBUTE SPACE

There are many ways to extend the two-attribute convex decomposition theorems in Section 3 to n-attribute decompositions. In this paper, we extend Theorem 1 to n attributes in a way which might be useful in the practical situations discussed later.

We partition X into X_i and $X_{\bar{i}}$, where $X_{\bar{i}} \equiv X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n$. When we consider $Y = X_i$ and $Z = X_{\bar{i}}$ in Theorem 1, all notation and definitions in the previous section are suffixed with i. The representation and its proof of n-attribute convex decomposition theorem requires some additional terminology and notation as shown in Farquhar [3]. First, we define the following function for $i = 1, \dots, n$,

$$G^m_{i,k_i} \equiv \sum_{k=1}^{m^*} \tilde{G}^m_{i(k_i,k)} G_i(x_i, x_{\bar{i}}^k), \tag{15}$$

where $\tilde{G}^m_{i(j,k)}$ is (j,k)-cofactor of G^m_i and $x_i \varepsilon X_i$, $x_{\bar{i}}^k \varepsilon X_{\bar{i}}$. The delta operator Δ is defined as follows. Suppose $X = X_{I_r} \times X_{\bar{I}_r}$ for some $1 \le r \le n$ and $I_r \subset \{1, \dots, n\}$. Let $y \varepsilon X_{\bar{I}_r}$ and $\alpha = \{\alpha_i : i \varepsilon I_r\}$, $\alpha_i \varepsilon \{1, \dots, m, *, \text{blank}\}$.

Then delta operator Δ is defined as

$$u(x_{I_r}^{\Delta\alpha}, y) \equiv \sum_{J \subset I_r} \{(-1)^b\, u(x_{i_1}^{\alpha_1}, \ldots, x_{i_r}^{\alpha_r}, y): a_j = 1 \text{ if } j \varepsilon J,$$

$$a_j = 0 \text{ and } \alpha_j = 0 \text{ if } j \notin J\}, \tag{16}$$

where $b = r + \sum_{j=1}^{r} a_j$.

We shall often omit attributes that are at the level x^0, when it will not be confusing. For instance, $u(x_1) = u(x_1, x_{\bar{1}}^0)$. The utility function is always scaled so that $u(x_1^0, \ldots, x_n^0) = 0$. From the definition of the delta operator and (1), $f_j(x_j^{\alpha_j}, x_J^{\Delta\alpha}, y)$ for all $j\varepsilon I_r$, $J=I_r-\{j\}$ are equal each other. Using the relation of $f_i(x_i^{\alpha_i}, x_{\bar{i}}) \equiv f_i(x_i^{\Delta\alpha}, x_{\bar{i}})$ for $i = 1, \ldots, n$, we can get the following notation

$$f_{I_r}^{\alpha}(y) \equiv f_i(x_{I_r}^{\Delta\alpha}, y) \text{ for all } i\varepsilon I_r. \tag{17}$$

The coefficient function $\Delta_{(I_r,\beta)}(y)$ for $I_r \subset \{1, \ldots, n\}$, $\beta = \{\beta_i: i\varepsilon I_r\}$ and $\beta_i \varepsilon \{1, \ldots, *\}$ is defined as

$$\Delta_{(I_r,\beta)}(y) \equiv \sum_{J \subset I_r} \{(-1)^b \prod_{i\varepsilon I_r} u(x_i^{a_i})\, f_{I_r}^{\beta}(y): a_j = \beta_j,\ \beta_j = * \text{ and } c_j = 0$$

$$\text{if } j\varepsilon J,\ a_j = * \text{ and } c_j = 1 \text{ if } j \notin J\}, \tag{18}$$

where $b = r + \sum_{j=1}^{r} c_j$ and $y \varepsilon X_{\bar{I}_r}$.

The coefficient function has the relation with (2) as follows.

PROPERTY 6.

(i) $\Delta_{(i,\beta_i)}(x_{\bar{i}}) = G_i(x_i^{\beta_i}, x_{\bar{i}})$ for $i = 1, \ldots, n$.

(ii) $\Delta_{(I_r,\beta)}(y) = u(x_i^*)\, \Delta_{(J,\beta)}(x_i^{\Delta\beta}, y) - u(x_i^{\beta_i})\, \Delta_{(J,\beta)}(x_i^{\Delta *}, y)$

for $i \varepsilon I_r$ and $J = I_r - \{i\}$.

(iii) $\Delta_{(J,\beta)}(y) = \sum_{K \subset J} \{(-1)^b\, G_i(x_K^{\Delta\beta}, y) \prod_{j \varepsilon I_r} u(x_j^{\alpha_j})$: $\alpha_j = \beta_j$, $\beta_j = *$ and

$a_j = 0$ if $j \varepsilon K$, $\alpha_j = *$ and $a_j = 1$ if $j \not\varepsilon K\}$,

where $b = r + \sum_{i=1}^{r} a_i$, $J = I_r + \{i\}$, $i \not\varepsilon I_r$ and $y \varepsilon X_{\bar{J}}$.

Proof. (i), (ii), and (iii) are easily obtained from (2) and (18). ■

THEOREM 3. Suppose that for $i \varepsilon N = \{1, \ldots, n\}$, m_i are nonnegative integers. For $i = 1, \ldots, n$, $X_i(CD_{m_i})X_{\bar{i}}$ if and only if

$$u(x_1, \ldots, x_n) = \sum_{I \subset N} \{c_I \prod_{i \varepsilon I} v_i(x_i)\}$$

$$+ \sum_{I \subset N} \{ \prod_{i \varepsilon I} d_i \sum_{j=1}^{m_i} G_{i,j}^{m_i}(x_i)[\Delta_{(I,\beta)} + V_I(x_J)]\}, \qquad (19)$$

where $V_I(X_J) \equiv \sum_{J \subset N-I} \{\Delta_{(I,\beta)}(x_J^{\Delta *}) \prod_{j \varepsilon J} v_i(x_i)\}$,

$c_I \equiv u(x_I^{\Delta *})$,

$d_i \equiv \dfrac{1}{|G_i^{m_i}|u(x_i^*)}$ for $i = 1, \ldots, n$,

$\beta = \{\beta_i : i \varepsilon I\}$ and $\beta_i \varepsilon \{1, \ldots, m_i, *\}$.

Proof. See appendix.

Decomposition form in Theorem 3 gives a wide range of utility functions on n-attribute space because it is possible to allow for the various orders of convex dependence among attributes. The order of convex dependence is the number of normalized conditional utility functions which must be evaluated to construct a multiattribute utility function. Therefore, Theorem 3 provides the general decomposition form which has m_i conditional utility functions on each X_i to be evaluated. Nahas [15] discussed the order of conditional utility function on each X_i when utility independence holds among attributes. In this paper, we show the relation among orders of convex dependence on each X_i, which is one extension of Nahas' discussion. As Property 4 holds with respect to the order of convex dependence between attributes, the following property holds with respect to the order of convex dependence in Theorem 3.

PROPERTY 7. When $X_i(CD_{m_i})X_{\bar{i}}$ for $i = 1, \ldots, n$, if $m_2, \ldots, m_n$ are arbitrary orders of convex dependence, the order m_1 must satisfy the following two inequalities.

$$\text{(i)} \quad \prod_{i=2}^{n} (m_i + 2) \geq m_1 + 1$$

$$\text{(ii)} \quad m_1 + 2 \geq \max \{a_2, \ldots, a_n\},$$

where $a_i = (m_1 + 1) \Big/ \prod_{\substack{j=2 \\ j \neq i}}^{n} (m_j + 2)$, $i = 2, \ldots, n$.

Proof: (i) When $m_2, \ldots, m_n$ are arbitrarily given, we can obtain the upperbound of m_1 by the following term in (19).

$$\prod_{i=1}^{n} d_i \sum_{j=1}^{m_i} G_{i,j}^{m_i}(x_i) \Delta_{(N,\beta)} \tag{20}$$

The upperbound of m_i is determined by the number of normalized conditional utility function on X_i included in (20). Then, it is sufficient to take into account the following term in (20).

$$d_i \sum_{j=1}^{m_i} G_{i,j}^{m_i}(x_i)\Delta_{(N,\beta)} \tag{21}$$

By Property 6 it is obvious that (21) is constructed by the linear combination of the following terms.

$$d_i \sum_{j=1}^{m_i} G_{i,j}^{m_i}(x_i)G_i(x_1^j, x_2^{\beta_2}, \ldots, x_n^{\beta_n}), \tag{22}$$

where $\beta_i \varepsilon \{0, 1, \ldots, m_i, *\}$, $i = 2, \ldots, n$.
Substituting (15) into (22), we have

$$d_i \sum_{j=1}^{m_i^*} G_i(x_i, x_{\bar{1}}^j) \sum_{k=1}^{m_i} \tilde{G}_{i(k,j)}^{m_i}\, G_1(x_1^k, x_2^{\beta_2}, \ldots, x_n^{\beta_n}). \tag{23}$$

Setting $x_{\bar{1}}^j = (x_2^{\beta_2}, \ldots, x_n^{\beta_n})$ in (23), we have

$$\frac{1}{u(x_1^*)} G_1(x_1, x_2^{\beta_2}, \ldots, x_n^{\beta_n}). \tag{24}$$

Then, the decomposition (19) includes $G_1(x_1, x_2^{\beta_2}, \ldots, x_n^{\beta_n})$, $\beta_i \varepsilon \{0, 1, \ldots, m_i, *\}$, $i = 1, \ldots, n$, that is, $\prod_{i=2}^{n} (m_i + 2)$ normalized conditional utility functions at most.

(ii) When $X_i(CD_{m_i})X_{\bar{i}}$, $i = 1, \ldots, n$, the orders $m_1, \ldots, m_n$ must satisfy the following inequalities by (i).

$$\prod_{\substack{j=1 \\ j \neq i}}^{n} (m_j + 2) \geq m_i + 1, \quad i = 1, \ldots, n$$

Then for m_1 we have

$$m_1 + 2 \geq \max \{a_2, \ldots, a_n\},$$

where $a_i = (m_i + 1)/\prod_{\substack{j=2 \\ j\neq i}}^{n} (m_j + 2), \quad i = 2, \ldots, n.$ ■

In some decision problems, utility independence may not hold in one or more attributes. In such cases the convex decomposition theorem may give a representation of the utility function. We illustrate how the convex decomposition theorem decomposes the utility function when $n = 3$.

When $m_1 = m_2 = m_3 = 0$ in (19), we have obviously

$$u(x_1, x_2, x_3) = \sum_{I \subset \{1,2,3\}} c_I \prod_{i\in I} v_i(x_i).$$

This decomposition is a multilinear utility function [11].

When m_2 and m_3 are arbitrary orders of convex dependence, we obtain the following inequalities from Property 7.

$$(m_2 + 2)(m_3 + 2) \geq m_1 + 1, \tag{25}$$

$$m_1 + 2 \geq \max \left\{\frac{m_2 + 1}{m_3 + 2}, \frac{m_3 + 1}{m_2 + 2}\right\} \tag{26}$$

When $m_2 = m_3 = 0$ in (25), that is, $X_2(CD_0)X_1X_3$ and $X_3(CD_0)X_1X_2$, X_1 is at most third-order convex dependent on X_2X_3. In this case the decomposition form in Theorem 3 is reduced to

$$\begin{aligned} u(x_1,x_2,x_3) = {} & \sum_{I \subset \{1,2,3\}} c_I \prod_{i\in I} v_i(x_i) \\ & + d_1 \sum_{i=1}^{m_1} G_{1,i}^{m_1}(x_1)[G_1(x_1^i,x_2^*,x_3^0)v_2(x_2) \\ & + G_1(x_1^i,x_2^0,x_3^*)v_3(x_3) + G_1(x_1^i,x_2^*,x_3^*)\, v_3(x_3)]. \end{aligned} \tag{27}$$

Therefore, we can construct (27) by evaluating one conditional utility function on X_2 and X_3, m_1 conditional utility functions on X_1, where $m_1 = 1$, 2, or 3, and constants. When $m_1 = 3$, that is, $X_1(CD_3)X_2X_3$, (27) is reduced to

$$u(x_1,x_2,x_3) = c_1v_1(x_1) + u(x_1,\overset{\Delta*}{x_2},\overset{0}{x_3})v_2(x_2)$$

$$+ u(x_1,\overset{0}{x_2},\overset{\Delta*}{x_3})v_3(x_3) + u(x_1,\overset{\Delta*}{x_2},\overset{\Delta*}{x_3})v_2(x_2)v_3(x_3). \quad (28)$$

This decomposition form is the same as the one which Keeney showed in [14] and Nahas discussed in [15] when $X_2(UI)X_1X_3$ and $X_3(UI)X_1X_2$. Keeney said nothing about what property holds between X_1 and X_2X_3 in this case. Convex dependence asserts that (28) holds if and only if $X_1(CD_3)X_2X_3$ as shown above. Moreover, from Property 7 (ii) convex dependence allows for $X_1(CD_2)X_2X_3$ or $X_1(CD_1)X_2X_3$ which are stronger conditions than $X_1(CD_3)X_2X_3$. In these cases, we could obtain decomposition forms easily as shown in (27) where $m_1 = 1$ and 2 are corresponding to $X_1(CD_1)X_2X_3$ and $X_1(CD_2)X_2X_3$, respectively.

5. SUMMARY

The concept of convex dependence is introduced for decomposing multiattribute utility functions. Convex dependence is based on normalized conditional utility functions. Since the order of convex dependence can be an arbitrary finite number, many different forms can be produced from the convex decomposition theorems. We have shown that the convex decompositions include the additive, multiplicative, multilinear and bilateral decompositions as special cases. A major advantage of the convex decompositions is that only single-attribute utility functions are used in the utility representations even for high-order convex dependent cases. Therefore, it is relatively easy

to assess the utility functions. Moreover, in the multiattribute case the orders of convex dependence among the attributes have much freedom even if the restrictions in Property 7 are taken into account. So even in the practical situations where utility independence, which is the 0-th order convex dependence, holds for all but one or two the attributes, the convex decompositions produce an appropriate representation.

Our approach is an approximation method based upon the exact grid model defined by Fishburn [7]. We note that Fishburn and Farquhar [8] recently established an axiomatic approach for a general exact grid model and provided a procedure for selecting a basis of normalized conditional utility functions.

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APPENDIX

To represent simply an arbitrary linear combination of normalized conditional utility functions, we define the following notation

$$C[v_{z_1}(y), \ldots, v_{z_n}(y)] \equiv \sum_{i=1}^{m} \theta_i \, v_{z_i}(y),$$

where $\sum_{i=1}^{m} \theta_i = 1$.

By using this notation, the following equations hold.

$$f(y,z) = f(y^*,z)C[v_{z^0}(y), v_z(y)], \tag{29a}$$

$$= f(y,z^*)C[v_{y^0}(z), v_y(z)], \tag{29b}$$

$$G(y,z) = G(y,z^*)C[v_{y^0}(z), v_y(z), v_{y^*}(z)], \tag{29c}$$

$$H(y,z) = H(y^*,z)C[v_{z^0}(y), v_z(y), v_{z^*}(y)]. \tag{29d}$$

Proof of Property 4: When $n = 0$, if $Y(CD_0)Z$, then $v_z(y) = v_{z^0}(y)$. Using (1), we have

$$f(y,z) = v_{z^0}(y)f(y^*,z). \tag{30}$$

Substituting (29b) into (30), we have

$$C[v_y(z), v_{y^0}(z)] = C[v_{y^*}(z), v_{y^0}(z)].$$

This concludes $Z(CD_1)Y$ at most.

When $n \geq 1$, if $Y(CD_n)Z$, then for distinct $z^0, z^1, \ldots, z^{n-1}, z^* \epsilon Z$

$$v_z(y) = [1 - \sum_{i=1}^{n^*} g_i(z)]v_{z^0}(y) + \sum_{i=1}^{n^*} g_i(z)v_{z^i}(y)$$

$$= \sum_{i=1}^{n^*} [v_{z^i}(y) - v_{z^0}(y)]g_i(z) + v_{z^0}(y). \tag{31}$$

By Property 2 we can select distinct $y^1, \ldots, y^n \epsilon Y$ and $z^1, \ldots, z^{n-1} \epsilon Z$ which make G_n a nonsingular matrix. Then, substituting these $y^1, \ldots, y^n \epsilon Y$ into (31), we have the following matrix equation,

$$G_n \underline{g} = \underline{v}, \tag{32}$$

where $\underline{g}$ and $\underline{v}$ are column vectors and these i-th elements are $g_i(z)$ and $v_z(y^i) - v_{z^0}(y^i)$, respectively.

Using $G(y,z)$, (32) is transformed into

$$\overline{G}\, \underline{g} = \underline{u}, \tag{33}$$

where $u(y^* z) \neq u(y^0,z)$ for all $z \epsilon Z$ from the previous assumption, and $(\overline{G})_{ij} = G(y^i,z^j)/[u(y^*,z^i) - u(y^0,z^j)]$, where $z^n = z^*$, and $\underline{u}$ is a column vector and its i-th element is $G(y^i,z)/[u(y^*,z) - u(y^0,z)]$.

Solving (33) for $g_i(z)$ $(i = 1, \ldots, n)$ and substituting these $g_i(z)$ into (31), we obtain

$$G(y,z) = \frac{1}{|G^n|} \sum_{i=1}^{n^*} G(y,z^i) \sum_{j=1}^{n} \widetilde{G}^n_{ji} G(y^j,z), \tag{34}$$

where G^n is nonsingular by Property 3.

By (29c) we have

$$G(y,z^*)C[v_{y^0}(z),\ v_y(z),\ v_{y^*}(z)]$$

$$= \frac{1}{|G^n|} \sum_{i=1}^{n^*} \widetilde{G}(y,z^i) \sum_{j=1}^{n} \widetilde{G}^n_{ji} G(y^j,\ z^*)C[v_{y^0}(z),\ v_{y^j}(z),\ v_{y^*}(z)]. \tag{35}$$

Summing up all the coefficients of $C[v_{y^0}(z),\ v_{y^j}(z),\ v_{y^*}(z)]$ for $j = 1, 2, \ldots, n$ in the right hand side of (35) yields

$$\frac{1}{|G^n|} \sum_{i=1}^{n^*} G(y,z^i) \sum_{j=1}^{n} \widetilde{G}^n_{ji} G(y^j,\ z^*) = G(y,z^*),$$

which implies

$$v_y(z) = C[v_{y0}(z),\ v_{y1}(z),\ \ldots,\ v_{yn}(z),\ v_{y*}(z)].$$

This concludes $Z(CD_{n+1})Y$ at most. ■

<u>Proof of Theorem 1</u>: Suppose $Y(CD_n)Z$, and (34) holds. Substituting (2) into the left hand side of (34) and solving it with respect to $u(y,z)$, then we have (11).

Conversely, suppose that (11) holds. By definition (2), it is obvious that $Y(CD_n)Z$. ■

<u>Proof of Theorem 2</u>: Suppose $Y(CD_n)Z$ and $Z(CD_n)Y$. Using Theorem 1, we get two equations,

$$u(y,z) = u(y^0,z) + u(y,z^0) + v(y)f(y^*,z) + c_y \sum_{i=1}^{n} G_i^n(y)G(y^i,z), \tag{36a}$$

and

$$u(y,z) = u(y^0,z) + u(y,z^0) + v(z)f(y,z^*) + c_z \sum_{i=1}^{n} H_i^n(z)H(y,z^i), \tag{36b}$$

where

$$G_i^n(y) \equiv \frac{1}{|G^n|} \sum_{k=1}^{n^*} \tilde{G}_{ik}^n G(y,z^k) \text{ and } H_i^n(z) \equiv \frac{1}{|H^n|} \sum_{k=1}^{n^*} \tilde{H}_{ik}^n H(y^k,z).$$

Substituting (36b) into $f(y^\alpha,z)$ for $\alpha\varepsilon\{1, 2, \ldots, n, *\}$, we have

$$f(y^\alpha,z) = v(z)f(y^\alpha,z^*) + c_z H(y^\alpha,z), \tag{37}$$

where we use $v(z^0) = 0$, $H(y,z^0) = 0$ and $H(y,z) = \sum_{i=1}^{n} H_i^n(z)H(y,z^i)$.

Substituting (36b) into $G(y^\alpha,z)$, we have

$$G(y^\alpha,z) = v(z)G(y^\alpha,z^*) + c_z \sum_{i=1}^{n} H_i^n(z)F(y^\alpha,z^i). \tag{38}$$

Substituting (37) and (38) into (36a), and using (2) and (3), we have

$$\begin{aligned} u(y,z) = {} & u(y^0,z) + u(y,z^0) + v(y)f(y^*,z) + v(z)f(y,z^*) \\ & - v(y)v(z)f(y^*,z^*) + c_y c_z \sum_{i=1}^{n} \sum_{j=1}^{n} G_i^n(y)H_j^n(z)F(y^i,z^j). \end{aligned} \tag{39}$$

We can assume that F^n is a nonsingular matrix because Property 5 holds. Considering next equation and transforming it, we obtain

$$v(y)f(y^*,z) + v(z)f(y,z^*) - v(y)v(z)f(y^*,z^*)$$

$$= |F^n|^{-1} \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} \tilde{F}^n_{ij}[v(y)f(y^*,z^i)f(y^j,z)$$

$$+ v(z)f(y^j,z^*)f(y,z^i) - v(y)v(z)f(y^j,z^*)f(y^*,z^i)]. \tag{40}$$

By definition (2) and (3), the following relation holds.

$$v(y)f(y^*,z^i)f(y^j,z) + v(z)f(y^j,z^*)f(y,z^i) - v(y)v(z)f(y^*,z^i)f(y^j,z^*)$$

$$= f(y,z^i)f(y^j,z) - c_v c_z G(y,z^i)H(y^j,z) \tag{41}$$

Substituting (40) and (41) into (39), we obtain

$$f(y,z) = \frac{1}{|F^n|} \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} \tilde{F}^n_{ij}[f(y,z^i)f(y^j,z) - c_y c_z G(y,z^i)H(y^j,z)]$$

$$+ c_y c_z \sum_{i=1}^{n} \sum_{j=1}^{n} G^n_i(y)H^n_j(z)F(y^i,z^j), \tag{42a}$$

$$f(y,z) = \frac{1}{|F^n|} \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} \tilde{F}^n_{ij} f(y,z^i)f(y^j,z) +$$

$$c_y c_z \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} [\frac{1}{|G^n H^n|} \sum_{k=1}^{n} \sum_{r=1}^{n} \tilde{G}^n_{ki}\tilde{H}^n_{rj} F(y^k,z^r) - \frac{\tilde{F}^n_{ij}}{|F^n|}]G(y,z^i)H(y^j,z). \tag{42b}$$

In (42a), setting $y = y^p$, $z = z^q$ for $p, q \varepsilon \{1, \ldots, n\}$, and solving it with respect to $c_y c_z F(y^p,z^q)$, and then substituting it into the following

$$\frac{c_y c_z}{|G^n H^n|} \sum_{k=1}^{n} \sum_{r=1}^{n} \tilde{G}^n_{ki}\tilde{H}^n_{rj} F(y^k,z^r) - \frac{F^n_{ij}}{|F^n|} c_y c_z$$

$$= \frac{\tilde{G}^n_{ni}\tilde{H}^n_{nj}}{|F^n G^n H^n|} [|F^n|f(y^n,z^n) - \sum_{p=1}^{n^*} \sum_{q=1}^{n^*} \tilde{F}^n_{qp} f(y^n,z^q)f(y^p,z^n)], \tag{43}$$

where we use the following relations

$$\sum_{p=1}^{n^*}\sum_{q=1}^{n^*}\tilde{F}^n_{qp} f(y^k,z^q)f(y^p,z^r) = |F^n| \sum_{q=1}^{n^*} \delta_{rq} f(y^k,z^q),$$

$$\sum_{k=1}^{n}\tilde{G}^n_{ki} G(y^k,z^q) = \delta_{iq}|G^n|, \text{ and } \sum_{r=1}^{n}\tilde{H}^n_{rj} H(y^p,z^r) = \delta_{jp}|H^n|,$$

where δ_{ij} denotes the Kronecker's delta.

Substituting (43) into (42b), then we have (12). Therefore, sufficient condition is proved.

Conversely, suppose that (12) holds, then we assume G^n, H^n, and F^n are nonsingular matrices. Substituting (3) and (29a) into (12), we have

$$f(y,z^*)C[v_{y0}(z),\ v_y(z)]$$

$$= \frac{1}{|F^n|}\sum_{i=1}^{n^*}\sum_{j=1}^{n^*}\tilde{F}^n_{ij} f(y,z^i)f(y^j,z)C[v_{y0}(z),\ v_{yj}(z)]$$

$$+ c\sum_{i=1}^{n^*}\sum_{j=1}^{n^*}\tilde{G}^n_{ni}\tilde{H}^n_{nj} G(y,z^i)\,\frac{v_{yj}(z) - v_{y0}(z)}{u(y^0,z^*)[u(y^i,z^*) - u(y^i,z^0)]}. \tag{44}$$

Summing up the coefficients of $C[v_{y0}(z),\ v_{yj}(z)]$, $v_{yj}(z)$ for $j = 0, 1, 2, \ldots, n, *$ and $v_{y0}(z)$ of the right hand side of (44), we have $f(y,z^*)$. Then, we conclude $Z(CD_n)Y$, and the same procedure for Y concludes $Y(CD_n)Z$. ■

<u>Proof of Theorem 3</u>: We can prove this theorem in the same way as Farquhar [3]. If $X_i(CD_{m_i})X_{\bar{i}}$ for $i = 1, \ldots, n$, then by Theorem 1, (15) and (18) the following equation holds.

$$u(x_1, \ldots, x_n) = u(x_i) + u(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$
$$+ v_i(x_i)f_i(x_1, \ldots, x_{i-1}, x_i^{\Delta *}, x_{i+1}, \ldots, x_n)$$
$$+ d_i\sum_{j=1}^{m_i} G^{m_i}_{i,j}(x_i)\Delta_{(i,\beta_i)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \tag{45}$$

If $i = 1$ in (45), then we have

$$u(x_1, \ldots, x_n) = u(x_1) + u(x_2, \ldots, x_n) + v_1(x_1)f_1(x_1^{\Delta *}, x_2, \ldots, x_n)$$

$$+ d_1 \sum_{j=1}^{m_1} G_{1,j}^{m_1}(x_1)\Delta_{(1,\beta_1)}(x_2, \ldots, x_n). \tag{46}$$

If $i = 2$ in (45), then we have

$$u(x_1, \ldots, x_n) = u(x_2) + u(x_1, x_3, \ldots, x_n) + v_2(x_2)f_2(x_1, x_2^{\Delta *}, x_3, \ldots, x_n)$$

$$+ d_2 \sum_{j=1}^{m_2} G_{2,j}^{m_2}(x_2)\Delta_{(2,\beta_2)}(x_1, x_3, \ldots, x_n). \tag{47}$$

We consider to substitute (46) into (47). First, we substitute (46) into the following

$$f_2(x_1, x_2^{\Delta c}, x_3, \ldots, x_n) = u(x_1, x_2^{\Delta c}, x_3, \ldots, x_n) - u(x_2)$$

$$= f_2(x_1^0, x_2^{\Delta c}, x_3, \ldots, x_n) + v_1(x_1)f_K^a(x_3, \ldots, x_n)$$

$$+ d_1 \sum_{j=1}^{m_1} G_{1,j}^{m_1}(x_1)\Delta_{(1,\beta_1)}(x_2^{\Delta c}, x_3, \ldots, x_n), \tag{48}$$

where $c \in \{0, 1, \ldots, m_2, *\}$, $K = \{1,2\}$, $a = \{a_1, a_2\}$, $a_1 = *$, $a_2 = c$ and we use the relation (17).

Secondly, we substitute (48) into the following

$$\Delta_{(2,\beta_2)}(x_1, x_3, \ldots, x_n)$$

$$= \Delta_{(2,\beta_2)}(x_1^0, x_3, \ldots, x_n) + v_1(x_1)\Delta_{(2,\beta_2)}(x_1^{\Delta *}, x_3, \ldots, x_n)$$

$$+ d_1 \sum_{j=1}^{m_1} G_{1,j}^{m_1}(x_1)\Delta_{(K,\beta)}(x_3, \ldots, x_n), \tag{49}$$

where $K = \{1,2\}$, and $\beta = \{\beta_1, \beta_2\}$.

From (46) we obtain the following

$$u(x_1, x_3 \ldots, x_n) = u(x_1) + u(x_3, \ldots, x_n) + v_1(x_1)f_1(x_1^{\Delta *}, x_2^0, x_3, \ldots, x_n)$$

$$+ d_1 \sum_{j=1}^{m1} G_{1,j}^{m1}(x_1)\Delta_{(1,\beta_1)}(x_2^0, x_3, \ldots, x_n). \qquad (50)$$

Substituting (48), (49), and (50) into (47), we have

$$u(x_1, \ldots, x_n) = u(x_1) + u(x_2) + u(x_3, \ldots, x_n)$$

$$+ v_1(x_1)f_1(x_1^{\Delta *}, x_2^0, x_3, \ldots, x_n)$$

$$+ v_2(x_2)f_2(x_1^0, x_2^{\Delta *}, x_3, \ldots, x_n)$$

$$+ v_1(x_1)v_2(x_2)f_K^a(x_3, \ldots, x_n)$$

$$+ d_1 \sum_{j=1}^{m1} G_{1,j}^{m1}(x_1) \left\{\Delta_{(1,\beta_1)}(x_2^0, x_3, \ldots, x_n) + v_2(x_2)\Delta_{(1,\beta_1)}(x_2^{\Delta *}, x_3, \ldots, x_n)\right\}$$

$$+ d_2 \sum_{j=1}^{m2} G_{2,j}^{m2}(x_2) \left\{\Delta_{(2,\beta_2)}(x_1^0, x_3, \ldots, x_n) + v_1(x_1)\Delta_{(2,\beta_2)}(x_1^{\Delta *}, x_3, \ldots, x_n)\right\}$$

$$+ d_1 d_2 \sum_{j=1}^{m1} \sum_{k=1}^{m2} G_{1,j}^{m1}(x_1)G_{2,k}^{m2}(x_2)\Delta_{(K,\beta)}(x_3, \ldots, x_n),$$

where $K = \{1,2\}$, $a = \{a_1, a_2\}$, $a_1 = *$, and $a_2 = *$.

This procedure is repeated for steps $i = 1, \ldots, n$. Hence, we have (19) by using Property 6 and the following relation

$$f_{I_r}^a = u(x_{I_r}^{\Delta *}) \text{ and } u(x_i) = u(x_i^{\Delta *})\, v_i(x_i) \text{ for } i = 1, \ldots, n,$$

where $I_r = \{i_1, \ldots, i_r\} \subset N$, $a = \{a_1, \ldots, a_r\}$ and $a_i = *$ for all i.

Converseely, if (19) holds, it is evidenly that $X_i(CD_{m_i})X_{\bar{i}}$ for $i = 1, \ldots, n$ by (29) and the property of convex combination. ■

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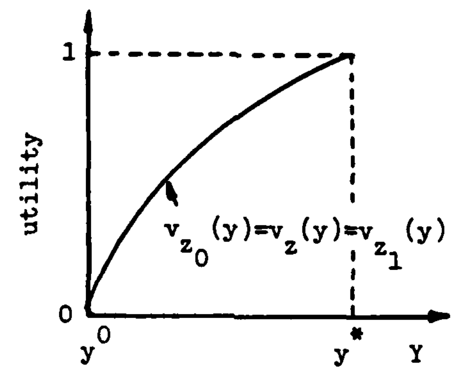
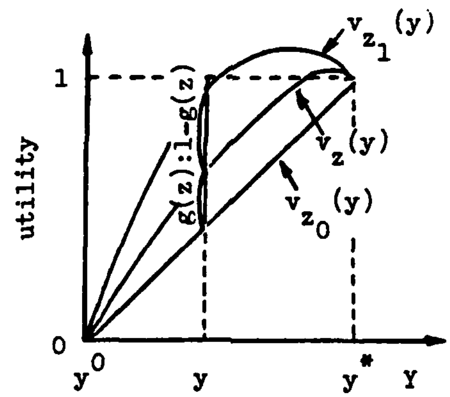


(a) $Y(CD_0)Z$

$v_z(y) = v_{z_0}(y)$ for all $z \in Z$



(b) $Y(CD_1)Z$

$v_z(y) = (1-g(z))v_{z_0}(y) + g(z)v_{z_1}(y)$ for all $z \in Z$

Figure 1. The relations among normalized conditional utility functions when the convex dependence holds.

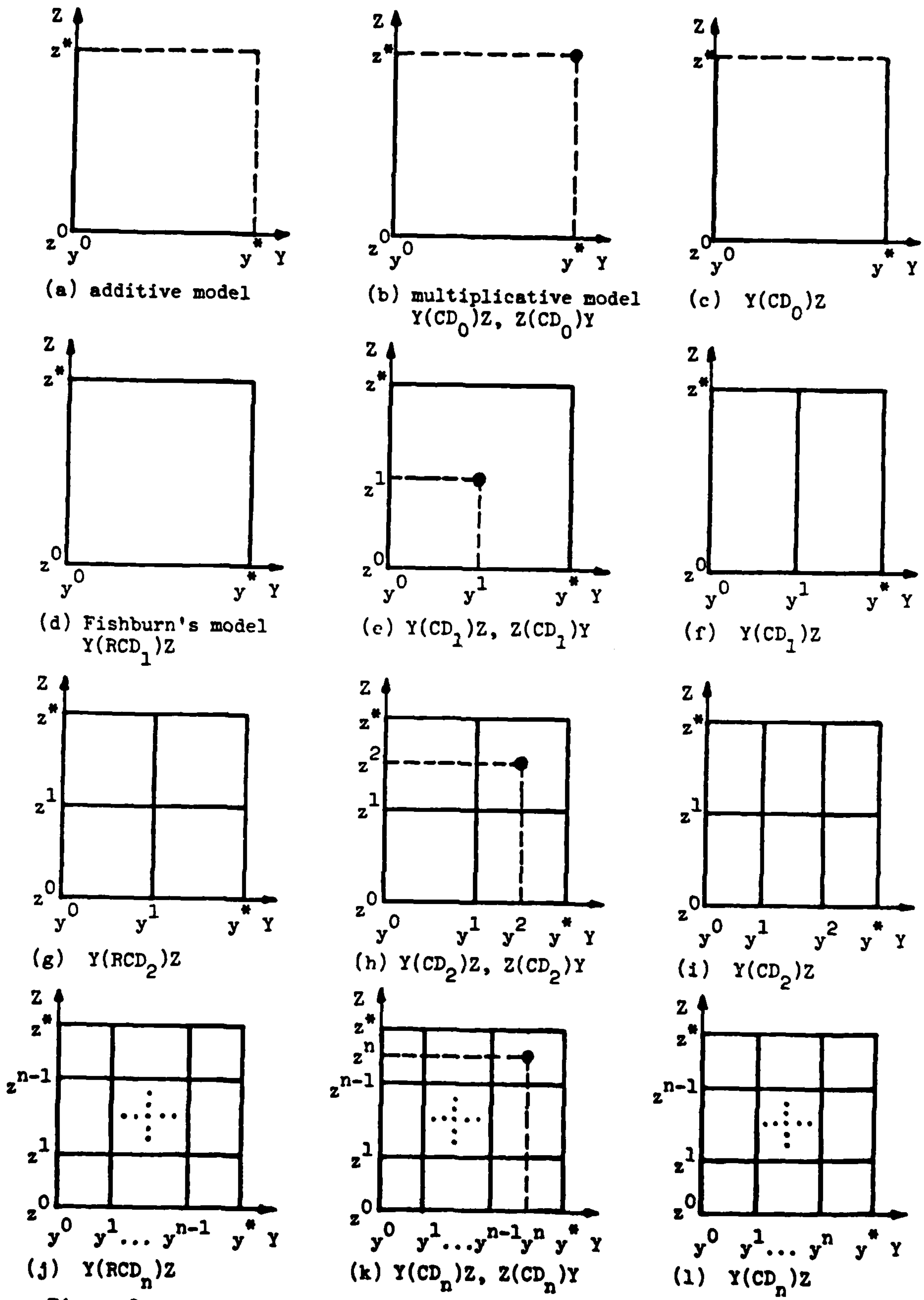


Figure 2. Assigning utilities for heavy shaded consequences completely specifies the utility function in the cases indicated.

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